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Multiple products of B-splines used in CAD system*

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Abstract The function upgrade of computer aided design (CAD) system requested that the multiple product of B-spline functions should be represented as a linear combination of some suitable (usually higher-degree) B-splines. In this paper, we apply the theory of spline space and discrete B-splines to deduce the representation of the coefficients of all terms of the linear combination, which can be directly applied to software coding in system development.

Keywords: computer aided design (CAD), NURBS, Marsden's identity.

Non-uniform rational B-spline (NURBS) was promulgated as the STEP standard[1] of industrial product data exchange by the International Organization for Standardization (ISO) in 1991, and CAD software companies all over the world also develop the surface design system based on the NURBS model[2]. Along with the industrial modernization and function upgrade of NURBS design system, we find that it is necessary to represent the multiple product of B-spline functions as a linear combination of some suitable B-splines, and this request becomes intensive day by day. For example, it is needed to integrate the products of three B-splines in computing the barycenter of the area surrounded by several planar B-spline curves, and to solve the derivative of the products of B-splines in computing the minimum distance between the two B-spline curves. Furthermore, the superposition of two NURBS curves demands reducing the fractions to a common denominator. If we can translate the multiplying of B-spline functions into an additive operation, then software coding in system development can be easily done. Moreover, it is necessary, without exception, to translate the multiplying of B-spline functions into an additive operation in the analysis of monotone curvature constraint condition for planar B-spline curves, in the least-squares approximation of B-spline curves, and in the offset approximation, the degree reduction approximation and the polynomial approximation of NURBS curves. However, till now, only Mørken[3] presented a product formula of two B-spline functions, and if we recursively apply Mørken's formula to solve the product problem of some B-spline functions, then it is difficult to apply the result to practice because the numbers of terms of summation formula will sharply increase and the corresponding notations will become very complicated. Therefore, we must seek for an explicit method which can directly express the product of $n\ (n \geqslant 2)$ B-spline functions as a linear combination of some suitable B-splines. It is not a simple and formal generalization of Mørken's formula, but requires a religious fine-draw of the transformation of spline's knot vector space based on the abstract theory of discrete B-spline, and even needs the innovation of computational technique.

In this paper, firstly, we generalize Marsden's identity representing the single power function as a linear combination of B-spline functions to the case of the product of multi-power functions; secondly, we deduce the degree formula and the knot vector formula for the products of $n\ (n \geqslant 2)$ B-spline functions; and finally, we obtain the coefficients formula of the expression translating the product into a summation, thereby changing the product of n B-spline functions into a linear combination of B-splines. The result is of practical importance in NURBS system development and engineering application.

1 Preliminaries and notations

Definition 1. Let k be a positive integer, and

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$N_{i,k,t}(x)$ given by the following de Boor-Cox recurrence formula, called the ith B-spline basis of order k (degree $k-1$), defined on a nondecreasing sequence $t=(t_i)$[2,4]

$$\begin{cases} N_{i,1,t}(x) = \begin{cases} 1, & \text{if } x \in [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases} \\ N_{i,k,t}(x) = \omega_{i,k}(x) N_{i,k-1,t}(x) \\ \qquad + (1-\omega_{i+1,k}(x)) N_{i+1,k-1,t}(x), \quad k \geqslant 2, \end{cases} \tag{1}$$

where

$$\omega_{i,k}(x) = \omega_{i,k,t}(x) = \begin{cases} (x-t_i)/(t_{i+k-1}-t_i), & \text{if } t_i < t_{i+k-1} \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Denote by $S_{k,t}$ the linear space spanned by these k order B-spline bases defined on the knot vector t. Obviously, $S_{k,t}$ consists of piecewise polynomials of degree $k-1$. In particular, the polynomial $p(x)=(x-y)^{k-1}$, where y is a real number, is in the space $S_{k,t}$, and its representation is given by Marsden's identity[4,5]:

$$(x-y)^{k-1} = \sum_i \psi_{i,k}(y) N_{i,k,t}(x),$$
$$\psi_{i,k}(y) = \psi_{i,k,t}(y) = (t_{i+1}-y)\cdots(t_{i+k-1}-y). \tag{3}$$

Similarly, an arbitrary spline function f in space $S_{k,t}$ can be expressed by the dual basis $\{\lambda_{i,k}\}$ as[3]:

$$\lambda_{i,k}f = \lambda_{i,k,t}f = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi_{i,k}^{(k-1-r)}(a_i) f^{(r)}(a_i) / (k-1)!, \tag{4}$$

where a_i is an arbitrary real number in $[t_i, t_{i+k}]$, and the right-hand derivative at t_i and the left-hand derivative at t_{i+k} are all supposed to be finite. If $t_i = t_{i+k}$, we use the convention $\lambda_{i,k}f = f(t_i)$.

Let $\tau=(\tau_j)$ be a subsequence of t, then $S_{k,\tau} \subseteq S_{k,t}$. The B-splines $\{N_{j,k,\tau}\}$ in $S_{k,\tau}$ are therefore linear combinations of the B-splines $\{N_{i,k,t}\}$, i.e.

$$N_{j,k,\tau} = \sum_i \alpha_{j,k,\tau,t}(i) N_{i,k,t}.$$

The coefficients $\{\alpha_{j,k,\tau,t}\}$ above are called discrete B-splines of order k[4,6]. Equivalently, an arbitrary spline $g = \sum_j c_j N_{j,k,\tau}$ in $S_{k,\tau}$ must satisfy

$$g = \sum_i d_i N_{i,k,t}, \quad d_i = \sum_j c_j \alpha_{j,k,\tau,t}(i). \tag{5}$$

Discrete B-splines are provided with a recurrence relation similar to the one in (1) for B-spline bases[6]:

$$\begin{aligned} \alpha_{j,k}(i) &= \alpha_{j,k,\tau,t}(i) \\ &= \omega_{j,k,\tau}(t_{i+k-1}) \alpha_{j,k-1}(i) \\ &\quad + (1-\omega_{j+1,k,\tau}(t_{i+k-1})) \alpha_{j+1,k-1}(i), \end{aligned} \tag{6}$$

where $\omega_{j,k,\tau}$ is given by (2), and $\alpha_{j,1}(i) = N_{j,1,\tau}(t_i)$.

2 A generalization of Marsden's identity for products of n B-spline functions

The key to establishing the B-spline representation of the product of n B-spline functions is expressing the polynomial $p(x) = \prod_{i=1}^{n} (x-y_i)^{k_i-1}$ by the B-spline bases $\{N_{i,k,t}\}$ associated with the knot vector t. Firstly, we give the following definition:

Definition 2. Let $k = \sum_{i=1}^{n} k_i - (n-1)$, here $k_i (i=1,2,\cdots,n)$ are positive integers. Suppose $P_1 = \{p_{11}, p_{12}, \cdots, p_{1,k_1-1}\}$ is a selection of $k_1 - 1$ integers from the set $I_{k-1} = \{1,2,\cdots,k-1\}$, and denote a selection of $k_2 - 1$ integers from the remaining $k-k_1$ integers (denote the set $I_{k-1} \setminus P_1$) by $P_2 = \{p_{21}, p_{22}, \cdots, p_{2,k_2-1}\}$. The rest may be deduced by analogy. Let $P_j = \{p_{j1}, p_{j2}, \cdots, p_{j,k_j-1}\}$ be a selection of $k_j - 1$ integers from the set $I_{k-1} \setminus P_1 \setminus P_2 \setminus \cdots \setminus P_{j-1} (j=2,3,\cdots,n)$. Here, there are $[k_{j+1} + k_{j+2} + \cdots + k_n - (n-j)]$ integers contained in the set I_{k-1}. For a given integer i, we define the corresponding knot vector

$$t^{P_j} = (\cdots, t_{i-1}, t_i, t_{i+p_{j1}}, t_{i+p_{j2}}, \cdots, t_{i+p_{j,k_j-1}}, t_{i+k}, t_{i+k+1}, \cdots), \quad j=1,2,\cdots,n, \tag{7}$$

with the digital subset P_j, and define the polynomial

$$\psi_{i,k_j,t^{P_j}}(y) = (t_{i+p_{j1}} - y)(t_{i+p_{j2}} - y) \cdots (t_{i+p_{j,k_j-1}} - y), \quad j=1,2,\cdots,n. \tag{8}$$

Furthermore, based on the selection according to the above-mentioned principle, we denote by $\prod = \prod_{k_1-1,\ \cdots,\ k_{j+1}-1,\ \cdots,\ k_{n-1}-1}^{k_1+k_2+\cdots+k_n-n,\ \cdots,\ k_{j+1}+k_{j+2}+\cdots+k_n-(n-j),\ \cdots,\ k_{n-1}+k_n-2}$ the digital set consisting of all the integer subsets $\{P_1, \cdots, P_{j+1}, \cdots, P_n\}$, then define its corresponding polynomial as

$$\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n) = \frac{\sum_{\{P_1,P_2,\cdots,P_n\}\in\prod}\left[\prod_{j=1}^{n}\psi_{i,k_j,t^{P_j}}(y_j)\right]}{\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{j+1}+k_{j+2}+\cdots+k_n-(n-j)}{k_{j+1}-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1}}. \quad (9)$$

We also denote by $F_j(j=1,2,\cdots,n)$ the polynomial formed by substituting the subscript "k_j-1" for "k_j" in the polynomial $\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)$. For example, $F_1=\psi_{i,k_1-1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)$.

Lemma 1. Suppose that $y_1,y_2,\cdots,y_n$ are all arbitrary real numbers, $a_j=(k_j-1)/(k-1)$, $\sum_{j=1}^{n}a_j=1$. Then we have

$$\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n) = \sum_{j=1}^{n}a_jF_j(t_{i+k-1}-y_j). \quad (10)$$

Proof. According to Definition 2, the polynomial $\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)$ corresponds to the digital set $\prod$, which implies that the digital sets I_{k-1} and $P_j(j=1,2,\cdots,n)$ can be uniquely confirmed. Furthermore, n B-spline functions corresponding to the digital set P_j can also be confirmed, and the associated order of each B-spline function is $k_j(j=1,2,\cdots,n)$, which satisfies $k=\sum_{j=1}^{n}k_j-(n-1)$. Similar to Definition 2, the digital sets I_{k-2} and $Q_{js}=\{q_{j1},q_{j2},\cdots,q_{j,k_{js}-1}\}(s=1,2,\cdots,n)$ corresponding to the polynomial $F_j(j=1,2,\cdots,n)$ can be uniquely confirmed along with the associated B-spline functions. The order of each B-spline function is $k_{js}(s=1,2,\cdots,n)$, and it satisfies $k-1=\sum_{s=1}^{n}k_{js}-(n-1)$, $k_{js}=k_s$, $j\neq s$, $s=1,2,\cdots,n$; $k_{jj}=k_j-1$.

By Definition 2, the procedure of forming the digital subsets $P_1,P_2,\cdots,P_n$ can be seen as selecting digital subsets $Q_{j1},Q_{j2},\cdots,Q_{jn}$ firstly and then inserting the element $(k-1)$ into the digital subset Q_{jj}. From this viewpoint, the digital subsets Q_{j1}, $Q_{j2},\cdots,Q_{jn}$ are equivalent to the part of the subsets $P_1,P_2,\cdots,P_n$ in which the corresponding subset P_j contains the element $(k-1)$. Therefore, the digital subsets $P_1,P_2,\cdots,P_n$ can be seen as the sum aggregates of all selected digital subsets $Q_{j1},Q_{j2},\cdots,Q_{jn}$, which correspond to the polynomials $F_j(j=1,2,\cdots,n)$.

According to (8), to each polynomial $F_j(j=1,2,\cdots,n)$, we have

$$\psi_{i,k_j,t^{P_j}}(y_j)=(t_{i+k-1}-y_j)\psi_{i,k_{jj},t^{Q_{jj}}}(y_j),$$

$$\psi_{i,k_s,t^{P_s}}(y_s)=\psi_{i,k_{js},t^{Q_{js}}}(y_s),$$

$$s\neq j,\ s=1,2,\cdots,n,$$

$$\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1} = \frac{k-1}{k_j-1}\binom{k_{j1}+k_{j2}+\cdots+k_{jn}-n}{k_{j1}-1}\cdots \cdot\binom{k_{j,n-1}+k_{jn}-2}{k_{j,n-1}-1}.$$

Thus, it can be seen that the value of the polynomial $\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)$ corresponding to the polynomial F_j $(j=1,2,\cdots,n)$ equals $\frac{k_j-1}{k-1}F_j(t_{i+k-1}-y_j)$.

From all the above analyses, we know that Lemma 1 is true.

Hence, the same arguments as those in Lemma 1 can reach the following lemma.

Lemma 2. Suppose $y_1,y_2,\cdots,y_n$ and a_j are the same as in Lemma 1. Then

$$\psi_{i-1,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n) = \sum_{j=1}^{n}a_jF_j(t_i-y_j). \quad (11)$$

Theorem 1. Suppose $y_1,y_2,\cdots,y_n$ are all arbitrary real numbers. Then we can obtain the generalization form of Marsden's identity:

$$\prod_{i=1}^{n}(x-y_i)^{k_i-1} = \sum_i\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)N_{i,k,t}(x). \quad (12)$$

Proof. Perform an induction on natural number k. Supposing first that $k=1$, i.e. $k_1=k_2=\cdots=k_n$

$=1$, we have $\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1, y_2, \cdots, y_n) = 1$. Suppose next that the result holds for splines of order $k-1$, then we must prove that it also holds for splines of order k. Denote by $f(x)$ the right-hand side of (12). Apply the usual recurrence relation (1) to B-spline bases, and then rearranging the order of the summation, we have

$$f(x) = \sum_i [\omega_{i,k}(x)\psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n) + (1-\omega_{i,k}(x))\psi_{i-1,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)] \cdot N_{i,k-1,t}(x).$$

From Lemma 1 and Lemma 2, substituting (10) and (11) into the above equation, and arranging the equation by simple algebraic operation, we can obtain

$$f(x) = \sum_i \sum_{j=1}^{n} a_j (x - y_j) F_j N_{i,k-1,t}(x).$$

Hence, by using the induction hypothesis and the identity $\sum_{j=1}^{n} a_j = 1$, we have

$$f(x) = \prod_{i=1}^{n} (x - y_i)^{k_i - 1}.$$

3 Explicit representation of the products of *n* B-spline functions

Let $f_i = \sum_{j_i} c_{j_i}^i N_{j_i,k_i,\tau_i}$ ($f_i \in S_{k_i,\tau_i}$, $i = 1, 2, \cdots, n$) be n given B-spline functions, where k_i and τ_i are its corresponding order and knot vector, respectively. If we want to get the explicit representation of the product of the B-spline functions, the first task is to construct the spline space $S_{k,t}$ which contains the product. Therefore, we give the following two lemmas to present the order and the knot vector of the space $S_{k,t}$, respectively.

Lemma 3. Suppose there are n B-spline functions being the same as the above, then the order of the product $f = \prod_{i=1}^{n} f_i$ is at least equal to

$$k = \sum_{i=1}^{n} k_i - (n-1). \tag{13}$$

Proof. Perform an induction on natural number n. Suppose first that $n=2$, and the conclusion is obvious[3]. Suppose next that the result holds for the product F of $n-1$ B-spline functions. Since $f = F f_n$, from the former conclusion, we can know that it also holds for n.

Definition 3. Let $L \in \{1, 2, \cdots, n\}$ be an integer and n be a positive integer. Suppose $P_j = \{p_1, p_2, \cdots, p_j\}$ is a selection of j integers from the set $I_L = \{1, 2, \cdots, L\}$, and denote the set consisting of the remaining $L-j$ integers by Q_{L-j}, that is, $Q_{L-j} = I_L \setminus P_j$, $j \leqslant L$, $j = 1, 2, \cdots, L$.

Applying the result of Ref. [3] and imitating the proof of Lemma 3, it is easy to get Lemma 4.

Lemma 4. Suppose that the n B-spline functions are the same as that in Lemma 3, and the knot y occurs with the multiplicity m_i ($m_i > 0$, $i = 1, 2, \cdots, n$) in the knot vector $\boldsymbol{\tau}_i$. Then the multiplicity m of the knot y in the knot vector t satisfies

$$m \geqslant \tilde{m} = \max\left(\sum_{j \in P_{n-1}} k_j + m_{Q_1} - (n-1) \right). \tag{14}$$

For the case that some of the multiplicities m_i of the knot y in the knot vector $\boldsymbol{\tau}_i$ equal zero, from Eq. (14) and Ref. [3], we can establish the required inequality; and if all of the multiplicities m_i ($i = 1, 2, \cdots, n$) equal zero, then the multiplicities of the knot y in the knot vector $\boldsymbol{t}$ must be zero.

The following theorem shows how the B-spline coefficients of the product are related to the B-spline coefficients of each factor.

Theorem 2. Suppose the n B-spline functions are the same as that in Lemma 3. Set $k = \sum_{i=1}^{n} k_i - (n-1)$ and construct the knot vector $\boldsymbol{t}$ as outlined above. Then $f = \prod_{i=1}^{n} f_i \in S_{k,t}$, and there exist coefficients d_i such that $f = \sum_i d_i N_{i,k,t}(x)$. Especially, for a given i, the knot vector $\boldsymbol{t}^{P_s}$ ($s = 1, 2, \cdots, n$) defined by (7) satisfies $\boldsymbol{\tau}_s \subseteq \boldsymbol{t}^{P_s}$, and d_i is given by

$$d_i = \frac{\sum_{\{P_1, P_2, \cdots, P_n\} \in \prod} \left(\prod_{s=1}^{n} \left(\sum_{j_s} c_{j_s}^s \alpha_{j_s, k_s, \tau_s, t^{P_s}}(i) \right) \right)}{\binom{k_1 + k_2 + \cdots + k_n - n}{k_1 - 1} \cdots \binom{k_{s+1} + k_{s+2} + \cdots + k_n - (n-s)}{k_{s+1} - 1} \cdots \binom{k_{n-1} + k_n - 2}{k_{n-1} - 1}}. \tag{15}$$

Proof. By the generalized Marsden's identity (12), we have

$$\prod_{j=1}^{n}(x-y_j)^{k_j-1}=\sum_i \psi_{i,k_1,k_2,\cdots,k_n,t}(y_1,y_2,\cdots,y_n)N_{i,k,t}(x)$$

$$=\frac{\sum_i \sum_{\{P_1,P_2,\cdots,P_n\}\in\prod}\left(\prod_{j=1}^{n}\psi_{i,k_j,t^{P_j}}(y_j)\right)N_{i,k,t}(x)}{\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{s+1}+k_{s+2}+\cdots+k_n-(n-s)}{k_{s+1}-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1}}. \tag{16}$$

Since $f_j(x)=(x-y_j)^{k_j-1}(j=1,2,\cdots,n)$, for any real number a_{ji} in $[t_i,t_{i+k}]$, we have

$$(y_j-a_{ji})^{k_j-1-r_j}=(-1)^{k_j-1-r_j}\frac{(k_j-1-r_j)!}{(k_j-1)!}f_j^{(r_j)}(a_{ji}),$$

$$r_j=0,1,\cdots,k_j-1.$$

Note that $f_j(x)$ is a polynomial of degree $k_j-1(j=1,2,\cdots,n)$. Hence, the order of the polynomial $f=\prod_{j=1}^{n}f_j$ whose degree is not more than $k-1$, is $k=\sum_{j=1}^{n}k_j-(n-1)$. Thus, according to "the polynomial theorem"[7] and "the Taylor expansions of n-variable"[8], after rearranging, Eq. (16) can be changed as

$$\prod_{j=1}^{n}f_j=\frac{\sum_i \sum_{\{P_1,P_2,\cdots,P_n\}\in\prod}\left(\prod_{j=1}^{n}\lambda_{i,k_j,t^{P_j}}(f_j)\right)N_{i,k,t}(x)}{\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{s+1}+k_{s+2}+\cdots+k_n-(n-s)}{k_{s+1}-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1}}. \tag{17}$$

In general, $f_j(x)$ is a k_j-1 degree piecewise polynomial in the spline space $S_{k_j,\tau_j}(j=1,2,\cdots,n)$. Because of the way in which the knot vector $\boldsymbol{t}$ was constructed we have $f(x)=\prod_{j=1}^{n}f_j(x)\in S_{k,t}$. From the uniqueness of the expression of the B-spline function, we can affirm that there exist unique coefficients d_i such that $f(x)=\sum_i d_iN_{i,k,t}(x)$. Especially, fix an integer i and consider an arbitrary nonempty subinterval (t_s,t_{s+1}) contained in the interval $[t_i,t_{i+k}]$. On this interval, the polynomials f_j and $g_j(j=1,2,\cdots,n)$ are equivalent to each other. Then $\prod_{j=1}^{n}g_j\in S_{k,t}$, and its B-spline coefficients can be obtained by (17). Comparing the B-spline coefficients of this polynomial with that of (17), and remembering that the B-splines are linearly independent, we have

$$d_i=\frac{\sum_{\{P_1,P_2,\cdots,P_n\}\in\prod}\prod_{j=1}^{n}\lambda_{i,k_j,t^{P_j}}(g_j)}{\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{s+1}+k_{s+2}+\cdots+k_n-(n-s)}{k_{s+1}-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1}}.$$

Note that the above expression holds for all nonempty subintervals (t_s,t_{s+1}) in $[t_i,t_{i+k}]$, we have

$$d_i=\frac{\sum_{\{P_1,P_2,\cdots,P_n\}\in\prod}\prod_{j=1}^{n}\lambda_{i,k_j,t^{P_j}}(f_j)}{\binom{k_1+k_2+\cdots+k_n-n}{k_1-1}\cdots\binom{k_{s+1}+k_{s+2}+\cdots+k_n-(n-s)}{k_{s+1}-1}\cdots\binom{k_{n-1}+k_n-2}{k_{n-1}-1}}. \tag{18}$$

In fact, Eq. (18) is just a disguised form of (15). According to (5), suppose $\boldsymbol{\tau}_1\subseteq \boldsymbol{t}^{P_1}$, then the number $\lambda_{i,k_1,t^{P_1}}(f_1)$ is just the ith B-spline coefficient of the B-spline function f_1 on the refined knot vector $\boldsymbol{t}^{P_1}$, so $\lambda_{i,k_1,t^{P_1}}(f_1)=\sum_{j_1}c_{j_1}^{1}\alpha_{j_1,k_1,\tau_1,t^{P_1}}(i)$, and similarly $\lambda_{i,k_s,t^{P_s}}(f_s)\ (s=2,3,\cdots,n)$. Applying the above conclusion to (18) yields (15).

It remains needing to prove that for a fixed integer i, $\boldsymbol{t}^{P_s}(s=1,2,\cdots,n)$ defined in (7) satisfy the relation $\boldsymbol{\tau}_s\subseteq \boldsymbol{t}^{P_s}$. Let y be a knot in the knot vector

$\boldsymbol{\tau}_s$, and let $m_1, m_2, \cdots, m_n, m, m_{P_1}, m_{P_2}, \cdots, m_{P_n}$ be the corresponding multiplicity in the knot vector $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_n, \boldsymbol{t}, \boldsymbol{t}^{P_1}, \boldsymbol{t}^{P_2}, \cdots, \boldsymbol{t}^{P_n}$, respectively. We must prove that $m_s \leqslant m_{P_s} (s = 1, 2, \cdots, n)$.

Supposing first that $m_s = 0$, recalling the construction of $\boldsymbol{t}^{P_s}$ defined by (7), we see that $m_{P_s} \geqslant 0$, namely, $m_s \leqslant m_{P_s}$.

If $m_s > 0$, there are three cases to be considered:

Case 1. If $y < t_{i+1}$ or $y > t_{i+k-1}$, recalling the construction of $\boldsymbol{t}^{P_s}$ defined by (7), we can find that $m_{P_s} = m$, and by Lemma 4, we have $m \geqslant m_s$.

Case 2. If $t_i < t_{i+1} \leqslant y \leqslant t_{i+k-1} < t_{i+k}$, we have $\sum_{s=1}^{n} m_{P_s} = m$. The worst case then occurs when $(m - m_{P_s})$ reaches the maximum, in other words, when $\max(m - m_{P_s}) = \sum_{j=1}^{s-1} k_j + \sum_{j=s+1}^{n} k_j - (n-1)$. Therefore, we have

$$m_{P_s} \geqslant m - \left[\sum_{j=1}^{s-1} k_j + \sum_{j=s+1}^{n} k_j - (n-1)\right].$$

By (14), the set I_n given by Definition 3 must include the selection condition of $P_{n-1} = \{1, \cdots, s-1, s+1, \cdots, n\}$, $Q_1 = \{s\}$, so we have $m_{P_s} \geqslant m_s$.

Case 3. If y equals the knots both within and outside the range $t_{i+1}, t_{i+2}, \cdots, t_{i+k-1}$ (e.g. $y = t_i = t_{i+1}$), a combination of the above two arguments can establish the required inequality.

The proof of Theorem 2 is completed.

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